# $\mathcal{PT}$-symmetric quantum field theories and the Langevin equation *)

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Many non-Hermitian but $\mathcal{PT}$-symmetric theories are known to have a real, positive spectrum, and for quantum-mechanical versions of these theories, there exists a consistent probabilistic interpretation. Since the action is complex for these theories, Monte Carlo methods do not apply. In this paper a field-theoretic method for numerical simulations of $\mathcal{PT}$-symmetric Hamiltonians is presented. The method is the complex Langevin equation, which has been used previously to study complex Hamiltonians in statistical physics and in Minkowski space. We compute the equal-time one-point and two-point Green's functions in zero and one dimension, where comparisons to known results can be made. The method should also be applicable in four-dimensional space-time. This approach may grant insight into the formulation of a probabilistic interpretation for path integrals in $\mathcal{PT}$-symmetric quantum field theories.

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## 1 Introduction

Traditionally, only theories with Hermitian Hamiltonians are studied in quantum mechanics and quantum field theory. Recently, however, it has been observed that $\mathcal{PT}$-symmetric Hamiltonians in one dimension often posses real eigenvalues as well as unitary time translation and conservation of probability [1–5]. Most of the work on $\mathcal{PT}$-symmetric Hamiltonians has been restricted to zero- and one-dimensional cases. This paper presents a method that should be applicable in higher dimensions and suggests that there is a real, Fokker–Planck probability underlying all of these theories. Further, it presents a numerical method for calculating the $k$-point Green's functions, $G_k$, of these theories. The results in this paper were presented previously and in more detail in [6].

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A $\mathcal{PT}$-symmetric theory that has been studied in the past is defined by the Euclidean Lagrangian

$$\mathcal{L}_{\mathrm{E}} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 - \frac{g}{N}(\mathrm{i}\phi)^N. \tag{1}$$

Note that $\mathcal{P}$ sends $\phi \to -\phi$, $\mathcal{T}$ sends $t \to -t$ and $\mathrm{i} \to -\mathrm{i}$, where $t$ is time.

Contours of integration and boundary conditions have been extensively studied in zero and one dimension [1, 7]. For arbitrary $N > 2$ the anti-Stokes' lines at the centers of the left and right wedges lie below the real axis at the angles

$$\theta_{\mathrm{left}} = -\pi + \frac{(N-2)\pi}{2N}, \quad \theta_{\mathrm{right}} = -\frac{(N-2)\pi}{2N}. \tag{2}$$

The opening angle of these wedges is $\pi/(2N)$. Similarly, for one-dimensional versions of Eq. (1) with $m = 0$ and for arbitrary $N > 2$, the anti-Stokes' lines at the centers of the left and right wedges lie below the real axis at the angles

$$\theta_{\mathrm{left}} = -\pi + \frac{N-2}{N+2}\frac{\pi}{2}, \quad \theta_{\mathrm{right}} = -\frac{N-2}{N+2}\frac{\pi}{2}. \tag{3}$$

The opening angle of these wedges is $2\pi/(N+2)$.

Consequently, expectation values for $\mathcal{PT}$-symmetric theories can be understood as path integrals that have been analytically continued in $N$. This analytic continuation deforms the contour from the real axis for the harmonic oscillator, $N = 2$, to contours in the complex-$\phi$ plane, whose endpoints lie in wedges where $\exp(-S(\phi))$ is damped as $|\phi| \to \infty$ so that the path integral converges. Defining the complex variable $\phi_{\mathrm{C}}$ to follow any contour whose endpoints lie in the appropriate wedges, $\mathcal{PT}$-symmetric expectation values of operators $A = A(\phi)$ are given by:

$$\frac{\langle 0|A|0\rangle}{\langle 0|0\rangle} = \frac{\int \mathrm{D}\phi_{\mathrm{C}}\ A(\phi_{\mathrm{C}})\ \mathrm{e}^{-S(\phi_c)}}{\int \mathrm{D}\phi_{\mathrm{C}}\ \mathrm{e}^{-S(\phi_{\mathrm{C}})}}, \tag{4}$$

where $S(\phi_{\mathrm{C}}) = \int \mathrm{d}^D X \mathcal{L}_{\mathrm{E}}[\phi_{\mathrm{C}}(X)]$ is the Euclidean space action.

Here, we use the complex Langevin method to calculate the equal-time one-point and two-point Green's functions for massless versions of Eq. (1) in zero and one dimension with $N = 3$ and $N = 4$. The results are in good agreement with those computed by numerical integration [1] and by variational methods [2, 7]. Reference [8] studies complex Hamiltonians in higher dimensions and also obtains accurate results. This suggests that the complex Langevin method is the robust numerical method needed for studying $\mathcal{PT}$-symmetric theories in higher dimensions. We believe our results represent a significant step towards a physical understanding of $\mathcal{PT}$-symmetric theories, and in the conclusion, we speculate connections may exist between the real Fokker–Planck probability in this paper and the probabilistic interpretations given in [4] and [5].

## 2 The Langevin method

Since many $\mathcal{PT}$-symmetric theories possess a real positive spectrum and a non-vanishing value for $G_1$, it has been speculated that $\mathcal{PT}$-symmetric theories could be used to describe a Higgs boson. A $-g\phi^4/4$ theory is especially interesting as a theory for the Higgs boson, because it has a dimensionless coupling constant and is asymptotically free [7]. Hence, a reliable numerical method is needed to compute expectation values in four-dimensional space-time, where physical quantities must be calculated.

One method for numerical calculations in quantum field theory involves the Langevin equation [9]

$$\frac{\partial\phi}{\partial\tau} = -\frac{\partial S(\phi)}{\partial\phi} + \eta(\tau), \tag{5}$$

where $\tau$ is an unphysical, Langevin time, $\partial S(\phi)/\partial\phi$ gives the equations of motion, and $\eta(\tau)$ is a stochastic variable. The function $\eta(\tau)$ is chosen to be a real, Gaussian random function, that satisfies the conditions $\langle\eta(\tau)\rangle = 0$ and $\langle\eta(\tau)\eta(\tau')\rangle = 2\delta(\tau - \tau')$, where the averaging is performed with respect to the appropriately normalized Gaussian probability distribution. It is well known that when $S(\phi)$ is real, the probability distribution, $P(\phi,\tau)$, associated with Eq. (5) is given by the Fokker–Planck equation [9].

For the case of real variables, calculating expectation values weighted by the Fokker–Planck probability and taking the limit $\tau \to \infty$ give results identical to those obtained by calculating expectation values by means of the path integral. This holds as long as the supersymmetric Fokker–Planck Hamiltonian, $H_{\rm FP}$, formed by taking $\partial S/\partial\phi$ as the superpotential, has a spectrum with positive real part and a ground state that is nondegenerate, but even when $S(\phi)$ is complex, these properties can still hold. Parisi and Klauder were the first to realize that these conditions allowed for the possibility of calculating expectation values for complex probability distributions [8, 10]. As we will explain, these criteria are the correct ones to test for convergence of expectation values of the Hamiltonians studied in this paper. This is also true for several other cases studied in Refs. [11–14]. This method was successful in several cases including statistical mechanics problems with complex chemical potentials and non-Hermitian Hamiltonians with complex eigenvalues[15].

We begin with the complex Fokker–Planck equation. Allowing $S(\phi)$ and $\phi$ to be complex, Eq.(5) can be divided into its real and imaginary parts and written as two coupled equations. If one assumes the noise is purely real and uses the methods of stochastic calculus to derive Ito's formula for two variables [9], one is lead to the complex Fokker–Planck equation

$$\begin{aligned}\frac{\partial P(\phi_{\rm R},\phi_{\rm I};\tau)}{\partial\tau} &= \left(\frac{\partial}{\partial\phi_{\rm R}}{\rm Re}\left[\frac{\partial S}{\partial\phi}\right] + \frac{\partial}{\partial\phi_{\rm I}}{\rm Im}\left[\frac{\partial S}{\partial\phi}\right] + \frac{\partial^2}{\partial\phi_{\rm R}^2}\right) P(\phi_{\rm R},\phi_{\rm I};\tau) \\ &\equiv O_{\rm FP}(\phi_{\rm R},\phi_{\rm I})P(\phi_{\rm R},\phi_{\rm I};\tau),\end{aligned} \tag{6}$$

where $\phi_{\rm R}$ and $\phi_{\rm I}$ are the real and imaginary parts of $\phi$, respectively, and $S = S(\phi_{\rm R} + {\rm i}\phi_{\rm I})$. Equation (6) defines a purely real probability in the complex-$\phi$ plane,

but apart from a few simple cases [13, 14], explicit constructions of $P(\phi_{\rm R}, \phi_{\rm I}; \tau)$ are unknown.

The average over the Langevin probability must be taken as an area integral in the complex plane given by

$$\frac{\langle 0|A|0\rangle_P}{\langle 0|0\rangle_P} = \frac{\int {\rm D}\phi_{\rm R}\ {\rm D}\phi_{\rm I}\ A(\phi_{\rm R}+{\rm i}\phi_{\rm I})P(\phi_{\rm R},\phi_{\rm I};\tau)}{\int {\rm D}\phi_{\rm R}\ {\rm D}\phi_{\rm I}\ P(\phi_{\rm R},\phi_{\rm I};\tau)}. \tag{7}$$

Notice that $A(\phi_{\rm R}+{\rm i}\phi_{\rm I})$ is an analytic function, but, in general, $P$ is not. Understanding how these expectation values correspond to those for the original theory is now complicated because in the limit $\tau \to \infty$ one must show how an area integral becomes a path integral and that a real, nonanalytic function, $P$, generates the complex, analytic function $\exp(-S(\phi))$. This can be achieved in a formal manner by following the approach introduced in Refs. [11, 12].

For the case of ${\rm i}g\phi^3/3$ in zero dimensions with $m = 0$, the path integral converges when $\exp(-S(\phi))$ is exponentially damped. Expressing the complex variable in polar coordinates, $\phi = r\,\exp({\rm i}\theta)$, the Stokes' regions that are traditionally chosen for $\mathcal{PT}$-symmetric theories are $-\pi < \theta < -2\pi/3$ and $-\pi/3 < \theta < 0$, as discussed in Sect. 1. Moreover, analytical calculations for these theories are most easily done along the contour where there is a pure exponential damping defined by $\theta = -5\pi/6$ and $\theta = -\pi/6$. However, any contour whose endpoints lie in the appropriate Stokes wedges is acceptable. For purposes of proving convergence for the Langevin expectation values, the easiest contour to use is $\phi = \phi_{\rm R} - {\rm i}b$, where $b$ is any finite constant. Along this contour, $\exp(-S(\phi)) = \exp(-{\rm i}g\phi^3/3) \sim$ (oscillatory term) $\times \exp(-g\phi_{\rm R}^2 b)$, and is therefore damped as $\phi_{\rm R} \to \pm\infty$.

The large-$\tau$ behavior of the expectation values given by Eq.(7) is discovered by shifting integration variables, $\phi_{\rm I} \to \phi_{\rm I} - b$. This shift does not affect the endpoints of integration and has a Jacobian of one. Any analytic function $A(\phi_{\rm R} - {\rm i}b + {\rm i}\phi_{\rm I})$ can be Taylor-expanded about the contour, $\phi_{\rm C} = \phi_{\rm R} - {\rm i}b$. This allows us to express the expectation values as

$$\frac{\langle 0|A|0\rangle_P}{\langle 0|0\rangle_P} = \frac{\int {\rm D}\phi_{\rm R}\ {\rm D}\phi_{\rm I}\ {\rm e}^{{\rm i}\chi}A(\phi_{\rm C})P(\phi_{\rm R},\phi_{\rm I}-b;\tau)}{\int {\rm D}\phi_{\rm R}\ {\rm D}\phi_{\rm I}\ P(\phi_{\rm R},\phi_{\rm I}-b;\tau)}, \tag{8}$$

where $\chi = \phi_{\rm I}\partial/\partial\phi_{\rm R}$. Integrating Eq. (8) by parts infinitely many times gives:

$$\frac{\langle 0|A|0\rangle_P}{\langle 0|0\rangle_P} = \frac{\int {\rm D}\phi_{\rm R}\ A(\phi_{\rm C})P_{\rm eff}(\phi_{\rm C},\tau)}{\int {\rm D}\phi_{\rm R}\ P_{\rm eff}(\phi_{\rm C},\tau)}, \tag{9}$$

where $P_{\rm eff}(\phi_{\rm C},\tau) = \int {\rm D}\phi_{\rm I}\ {\rm e}^{-{\rm i}\chi}P(\phi_{\rm R},\phi_{\rm I}-b;\tau)$. We assume that $P$ vanishes at infinity rapidly enough so that all of the boundary terms from the integration by parts are zero. (In the denominator of Eq. (8) $\exp(-{\rm i}\chi)$ can be introduced for free because all but the zeroth-order term in $\chi$ integrate to zero.) Note that $P_{\rm eff}$ is an analytic function of $\phi_{\rm C} = \phi_{\rm R} - {\rm i}b$, not a function of $\phi_{\rm R}$ and $b$ separately. We see this by using $\exp(-{\rm i}\chi)P(\phi_{\rm R},\phi_{\rm I}-b;\tau) = P(\phi_{\rm R}-{\rm i}\phi_{\rm I},\phi_{\rm I}-b;\tau)$ and then shifting the integration variable $\phi_{\rm I} \to \phi_{\rm I} + b$, so that $P_{\rm eff} = \int {\rm D}\phi_{\rm I}\ P(\phi_{\rm C}-{\rm i}\phi_{\rm I},\phi_{\rm I};\tau)$. As a

result, Eq. (9) can be equivalently written as an integral along the contour defined by $\phi_{\rm C}$.

We now derive a pseudo Fokker–Planck equation for $P_{\rm eff}(\phi_{\rm C},\tau)$. By definition,

$$\frac{\partial P_{\rm eff}(\phi_{\rm C},\tau)}{\partial\tau} = \int {\rm D}\phi_{\rm I}\ {\rm e}^{-{\rm i}\chi}\frac{\partial P(\phi_{\rm R},\phi_{\rm I}-b;\tau)}{\partial\tau}$$
$$= \int {\rm D}\phi_{\rm I}\ O_{\rm FP}^{\rm eff}{\rm e}^{-{\rm i}\chi}P(\phi_{\rm R},\phi_{\rm I}-b;\tau), \qquad (10)$$

where we have used Eq. (6) and $O_{\rm FP}^{\rm eff}\equiv\exp(-{\rm i}\chi)O_{\rm FP}(\phi_{\rm R},\phi_{\rm I}-b)\exp({\rm i}\chi)$. Using the relations ${\rm e}^{-{\rm i}\chi}\partial{\rm e}^{{\rm i}\chi}/\partial\phi_{\rm R}=\partial/\partial\phi_{\rm R}$, ${\rm e}^{-{\rm i}\chi}\partial{\rm e}^{{\rm i}\chi}/\partial\phi_{\rm I}=\partial/\partial\phi_{\rm I}+{\rm i}\partial/\partial\phi_{\rm R}$, and ${\rm e}^{-{\rm i}\chi}F(\phi_{\rm C}+{\rm i}\phi_{\rm I}){\rm e}^{{\rm i}\chi}=F(\phi_{\rm C})$, it is straightforward to show that

$$O_{\rm FP}^{\rm eff}=\frac{\partial}{\partial\phi_{\rm R}}\left(\frac{\partial}{\partial\phi_{\rm R}}+\frac{\partial S(\phi_{\rm R}-{\rm i}b)}{\partial\phi_{\rm R}}\right)+\frac{\partial}{\partial\phi_{\rm I}}{\rm e}^{-{\rm i}\chi}{\rm Im}[\frac{\partial S(\phi_{\rm R}-{\rm i}b+{\rm i}\phi_{\rm I})}{\partial\phi}]{\rm e}^{{\rm i}\chi}. \qquad (11)$$

The last term of Eq. (11) is a total derivative in $\phi_{\rm I}$, and therefore, disappears from the right side of Eq. (10), again assuming $P$ vanishes rapidly at infinity. Since the remaining terms of $O_{\rm FP}^{\rm eff}$ do not depend on $\phi_{\rm I}$, they can be pulled out in front of the integral over ${\rm D}\phi_{\rm I}$. Using the fact that $\partial/\partial\phi_{\rm R}=\partial/\partial\phi_{\rm C}$ on an analytic function of $\phi_{\rm C}$, Eq. 10) becomes analogous to the Fokker–Planck equation restricted to the real axis, except $\phi=\phi_{\rm C}$ and $P(\phi,\tau)=P_{\rm eff}(\phi_{\rm C},\tau)$

$$\frac{\partial P_{\rm eff}(\phi_{\rm C},\tau)}{\partial\tau}=\frac{\partial}{\partial\phi_{\rm C}}\left(\frac{\partial}{\partial\phi_{\rm C}}+\frac{\partial S(\phi_{\rm C})}{\partial\phi_{\rm C}}\right)P_{\rm eff}(\phi_{\rm C},\tau). \qquad (12)$$

This is a pseudo-Fokker–Planck equation that defines a complex function, $P_{\rm eff}$.

For cases where $N>3$ in Eq. (1), a similar derivation gives the same result. The only subtlety is in choosing the correct contour. Thus, the problem of understanding the $\tau\to\infty$ behavior of expectation values has been reduced to one, that is formally identical to that for real variables. We now use the methods of Parisi and Sourlas [16], who first discovered the hidden supersymmetry in classical stochastic equations.

If we express Eq. (12) in terms of $p(\phi_{\rm C},\tau)\equiv P_{\rm eff}(\phi_{\rm C},\tau)\exp(S(\phi)/2)$, we obtain the Schrödinger equation

$$\frac{-\partial p(\phi_{\rm C},\tau)}{\partial\tau}=H_{\rm FP}p(\phi_{\rm C},\tau)\equiv\left(-\frac{\partial}{\partial\phi_{\rm C}}+\frac{1}{2}\frac{\partial S}{\partial\phi_{\rm C}}\right)\left(\frac{\partial}{\partial\phi_{\rm C}}+\frac{1}{2}\frac{\partial S}{\partial\phi_{\rm C}}\right)p(\phi_{\rm C},\tau). \qquad (13)$$

As claimed, $H_{\rm FP}$ is the supersymmetric Hamiltonian formed from the superpotential $\partial S/\partial\phi_{\rm C}$. Since $S$ is $\mathcal{PT}$-symmetric, $\partial S/\partial\phi_{\rm C}$ is anti-$\mathcal{PT}$-symmetric, and $H_{\rm FP}$ is $\mathcal{PT}$-symmetric. If the time-independent version of Eq. (13), $H_{\rm FP}\Psi_k^{\rm FP}(\phi_{\rm C})=\lambda_k\Psi_k^{\rm FP}(\phi_{\rm C})$, is well posed and the eigenfunctions $\Psi_k^{\rm FP}(\phi_{\rm C})$ are complete, then $p(\phi_{\rm C},\tau)=\sum_{k=0}^{\infty}a_k\Psi_k^{\rm FP}(\phi_{\rm C}){\rm e}^{-\lambda_k\tau}$. Note that $\Psi_0^{\rm FP}(\phi_{\rm C})\equiv\exp(-S(\phi_{\rm C})/2)$ is an eigenfunction of $H_{\rm FP}$ with $\lambda_0=0$. Therefore, the wave-function expansion becomes

$$p(\phi_{\rm C},\tau)=C{\rm e}^{-(1/2)S(\phi_{\rm C})}+\sum_{k=1}^{\infty}a_k\Psi_k^{\rm FP}(\phi_{\rm C}){\rm e}^{-\lambda_k\tau}. \qquad (14)$$

Moreover, if the spectrum of $H_{\rm FP}$ is such that $\mathrm{Re}[\lambda_k] > 0,\ k > 0$, it follows that $p(\phi_{\rm C},\tau) \to C\mathrm{e}^{-(1/2)S(\phi_{\rm C})},\ \tau \to \infty$. The $\tau$ dependence of $p(\phi_{\rm C},\tau)$ has disappeared in this limit. That implies $\mathrm{d}P_{\rm eff}/\mathrm{d}\tau = 0$ and signals that the system has reached equilibrium. Expressing $P_{\rm eff}(\phi_{\rm C},\tau)$ in terms of $p(\phi_{\rm C},\tau)$ and taking the limit $\tau \to \infty$ gives $\exp(-S(\phi))$. As a result, Langevin expectation values are shown to converge to the right side of Eq. (4) as $\tau \to \infty$. This result holds as long as the ground state is nondegenerate. There is no evidence that $\mathcal{PT}$-symmetric theories possess a degenerate ground state, so for the purposes of this paper, we shall not consider this a possibility.

Thus, if the supersymmetric Hamiltonian $H_{\rm FP}$, formed from the superpotential, $\partial S/\partial\phi_{\rm C}$, has a spectrum with a real part, that is always greater than or equal to zero, the Langevin method should work as a calculational procedure. To summarize, we have shown that $P_{\rm eff}(\phi_{\rm C},\tau) \to \mathrm{e}^{-S(\phi_{\rm C})},\ \tau \to \infty$, and

$$\frac{\int \mathrm{D}\phi_{\rm R}\ \mathrm{D}\phi_{\rm I}\ A(\phi_{\rm R}+\mathrm{i}\phi_{\rm I})P(\phi_{\rm R},\phi_{\rm I};\tau)}{\int \mathrm{D}\phi_{\rm R}\ \mathrm{D}\phi_{\rm I}\ P(\phi_{\rm R},\phi_{\rm I};\tau)} = \frac{\int_{-\infty-\mathrm{i}b}^{\infty-\mathrm{i}b}\mathrm{D}\phi_{\rm C}\ A(\phi_{\rm C})P_{\rm eff}(\phi_{\rm C},\tau)}{\int_{-\infty-\mathrm{i}b}^{\infty-\mathrm{i}b}\mathrm{D}\phi_{\rm C}\ P_{\rm eff}(\phi_{\rm C},\tau)}$$
$$\to \frac{\int_{-\infty-\mathrm{i}b}^{\infty-\mathrm{i}b}\mathrm{D}\phi_{\rm C}\ A(\phi_{\rm C})\mathrm{e}^{-S(\phi_{\rm C})}}{\int_{-\infty-\mathrm{i}b}^{\infty-\mathrm{i}b}\mathrm{D}\phi_{\rm C}\ \mathrm{e}^{-S(\phi_{\rm C})}},\ \tau\to\infty\,, \tag{15}$$

follow, if $H_{\rm FP}$ has a nondegenerate ground state, wave functions that are complete, and a spectrum with positive real part.

## 3 Numerical methods and results

Dorey *et al.* [3] showed that $\mathcal{PT}$-symmetric Hamiltonians of the form $\mathcal{H} = -(\partial\phi)^2 - (\mathrm{i}\phi)^{2M} - \alpha(\mathrm{i}\phi)^{M-1}$, where $M$ and $\alpha$ are real and boundary conditions have been chosen as in Sect. 1, have a positive real spectrum if the conditions $\alpha < M$ and $M \geq 1$ are both satisfied [3]. This proves (for $\alpha = 0$, $M = N/2$) that massless versions of Eq. (1) have a positive real spectrum. In zero-dimensional studies of Eq. (1) with $m = 0$, Eq. (13) gives $H_{\rm FP} = -\partial^2/\partial\phi_{\rm C}^2 - \frac{1}{2}g(N-1)(\mathrm{i}\phi_{\rm C})^{N-2} - \frac{1}{4}g^2(\mathrm{i}\phi_{\rm C})^{2(N-1)}$, where the contour $\phi_{\rm C}$ is within the Stokes' wedges, explained in Sect. 1 and used by Dorey *et al.*. Making the change of variables $\phi \to (2/g)^{1/N}\phi$, the Hamiltonians studied in this paper become the same as Dorey *et al.*'s Hamiltonians with $\alpha = N-1$ and $M = N-1$. Thus, $\alpha = M$, and as explained in Ref. [3], this implies that the $H_{\rm FP}$ for these $\mathcal{PT}$-symmetric theories has one zero eigenvalue, which we have already demonstrated, and that all of the remaining eigenvalues are real and positive. Consequently, the spectrum is real and non-negative holds for $N \geq 2$, and this implies that the complex Langevin method will work.

Denoting the equations of motion by the function $F(\phi(j))$, the simplest discretization of this is Euler's method $\phi(j+1) = \phi(j) + h(F(\phi(j)) + \eta(j)) = \phi(j) + \epsilon^2 F(\phi(j)) + \epsilon\eta'(j)$, where $j$ is an index for a Langevin-time step, $h$ is the spacing in Langevin time, $\epsilon = \sqrt{h}$, and the $h$-dependence of $\eta(\tau)$ has been made explicit $\eta'(j) = \epsilon\eta(j)$ and $\langle\eta'(j)\eta'(k)\rangle = 2\delta_{j,k}$. This form for $\eta'(\tau)$ follows from the normalization of the Gaussian probability distribution, which implies $\eta^2(\tau) \sim \delta(0) \sim 1/h$

on a lattice. For this and the second-order algorithm given below, there are numerical instabilities for large values of $\phi$. For $-g\phi^4/4$, it was necessary to restrict the absolute magnitude of $\phi$ in order to avoid these instabilities. Limiting values were obtained by fitting the data with second-degree and third-degree polynomials in $\epsilon$. Errors are calculated by collecting the simulation data in bins of a given size and computing the standard deviation of the means of the bins. The maximum error as a function of bin size is taken to be the error for the simulation. In Table 1 the numerical results, obtained using Euler's method, are compared with exact values given in Ref. [7]. The parts of the one-point and two-point disconnected Green's functions that are known to vanish (e.g. $\mathrm{Re}[G_1]$) have errors larger than their magnitude as $\epsilon \to 0$.

Table 1. Numerically determined values of $\mathrm{i}G_1 = \mathrm{i}\langle 0|\phi|0\rangle/\langle 0|0\rangle$ and $-G_2 = -\langle 0|\phi^2|0\rangle/\langle 0|0\rangle$ using Euler's method and a second-order method for zero-dimensional $\mathrm{i}g\phi^3/3$ and $-g\phi^4/4$ theories with $g = 1/2$. These limiting values were determined by fitting the simulated data to polynomials of second and third degree in $\epsilon$. For the second-order algorithm, the term linear in $\epsilon$ is set to zero. Exact results are listed to four significant digits in the first column. Note that the values listed for the second-order algorithm are indeed more precise than the results using Euler's method.

| $N$ | $\mathrm{i}G_1^{\mathrm{exact}}$ | $\mathrm{i}G_1^{\mathrm{Euler}}$ | $\mathrm{i}G_1^{\mathrm{2nd\ order}}$ | $-G_2^{\mathrm{exact}}$ | $-G_2^{\mathrm{Euler}}$ | $-G_2^{\mathrm{2nd\ order}}$ |
|---|---|---|---|---|---|---|
| 3 | 0.9185 | 0.9198(14) | 0.9194(7) | 0 | – | – |
| 4 | 1.163 | 1.166(3) | 1.164(1) | 0.9560 | 0.9623(51) | 0.9595(14) |

Since Euler's method converges linearly in $\epsilon$ as $\epsilon \to 0$, a more accurate method of second-order in $\epsilon$ is desirable. A second-order Runge–Kutta algorithm, that led to good results in previous studies and was first developed in Ref. [17], is

$$\begin{aligned} \tilde{\phi}(j) &= \phi(j) + \epsilon^2 F(\phi(j)) + \epsilon\eta'(j)\,, \\ \phi(j+1) &= \phi(j) + sfrac12\epsilon^2(F(\phi(j)) + F(\tilde{\phi}(j))) + \epsilon\eta'(j). \end{aligned} \tag{16}$$

In our studies this method is numericall more stable than Euler's method, and therefore allows the inclusion of more data. Limiting values were obtained by fitting the data with second-degree and third-degree polynomials in $\epsilon$ with the linear term set equal to zero. The results obtained using this algorithm are compared with exact values and the result of Euler's method in Table 1.

Now we apply the complex Langevin method to massless versions of Eq. (1) in one dimension with $N = 3$ and $N = 4$. The potentials are the same as those discussed in zero dimensions, but now there is a second derivative in physical time, $-\partial^2\phi(t,\tau)/\partial t^2$, present in all of the equations, and $\eta(\tau)$ becomes $t$ dependent. Thus, the eigenvalue problem for $H_{\mathrm{FP}}$ becomes a partial differential equation. The spectra of eigenvalue problems for $\mathcal{PT}$-symmetric partial differential equations has not been studied, and we do not have a proof that they are real. But, following the examples of zero-dimensional theories, we assume the spectrum of $H_{\mathrm{FP}}$ has a positive real part. The results of our simulations support this assumption.

Table 2. Numerically determined values of $\mathrm{i}G_1 = \mathrm{i}\langle 0|\phi|0\rangle/\langle 0|0\rangle$ and $-G_2 = -\langle 0|\phi^2|0\rangle/\langle 0|0\rangle$ using Euler's method and a second-order method for one-dimensional $-g(\mathrm{i}\phi)^N/N$ theories with $N = 3$ and $N = 4$. In order to compare our results with Lagrangians used in previous studies, we have set $g = N/2$. These continuum-limit values were determined by fitting the simulated data to third-degree polynomials in $\epsilon$ for a given $a$ (for the second-order algorithm, the term linear in $\epsilon$ is set to zero), and then fitting the results of those fits with second-degree polynomials in $a^2$. Exact results obtained by direct numerical integration of the quantum-mechanical problem are listed to four significant digits in the first column. The exact value of $-G_2$ for $N = 4$ has never been directly calculated, so instead we list the results of the variational calculation given in Ref. [2]. Note that the values listed for the second-order algorithm are indeed more precise than the results using Euler's method.

| $N$ | $\mathrm{i}G_1^{\mathrm{exact}}$ | $\mathrm{i}G_1^{\mathrm{Euler}}$ | $\mathrm{i}G_1^{\mathrm{2nd\ order}}$ | $-G_2^{\mathrm{var}}$ | $-G_2^{\mathrm{Euler}}$ | $-G_2^{\mathrm{2nd\ order}}$ |
|---|---|---|---|---|---|---|
| 3 | 0.5901 | 0.5890(5) | 0.5898(2) | 0 | – | – |
| 4 | 0.8669 | 0.8654(6) | 0.8670(3) | 0.5182 | 0.5171(9) | 0.5183(4) |

These one-dimensional theories are more computationally exacting because they give rise to Langevin equations that are partial differential equations. Consequently, the second-order algorithm in Eq. (16) is useful. For numerical simulations of one-dimensional theories, we have to introduce a physical-time lattice and $-\partial^2\phi(t,\tau)/\partial t^2$ becomes $-[\phi(j,l+1) - 2\phi(j,l) + \phi(j,l-1)]/a^2$, where $l$ is an index for real time and $a$ is the spacing in physical time. The algorithms are the same as for zero dimensions except they now contain the physical-time derivative as part of $F$ and the explicit lattice dependence of $\eta(t,\tau)$ is such that $\eta'(j,l) = \sqrt{ha}\eta(j,l)$ and $\langle\eta'(j,l)\eta'(k,m)\rangle = 2\delta_{j,k}\delta_{l,m}$. Further, in these algorithms there is always an $\epsilon^2$ associated with the physical-time derivative, and thus, the simulation is unstable unless $\epsilon \ll a$. However, there are no instabilities encountered for large $|\phi|$. For fixed values of $a$, we compute at various values of $\epsilon$ and take the limit $\epsilon \to 0$, giving the expectation values as a function of $a$. We then take $a \to 0$ and obtain the expectation values in the continuum limit. In Table 2 the numerical results for the expectation values are compared with numerically integrated quantum-mechanical values given in Ref. [2]. We have chosen $g$ such that Eq. (1) agrees with the Hamiltonians in Ref. [2].

## 4 Conclusions and speculations

This paper and Ref. [7] have shown that $\mathcal{PT}$-symmetric theories are amenable to the methods of quantum field theory. Here we show how a numerical method based on the complex Langevin equation, can be used to obtain precise results. We believe that this numerical method can be applied to higher-dimensional theories. This work represents an important step towards a test of the physical applicability of $\mathcal{PT}$-symmetric Hamiltonians because it should allow one to compute the mass of the Higgs particle in four-dimensional versions of these theories. Before this can be done, however, renormalization must be thoroughly understood.

Perhaps the most important implication for $\mathcal{PT}$-symmetric theories is the implicit probability distribution defined by Eq. (6). This study shows that expectation values in $\mathcal{PT}$-symmetric quantum field theory can be interpreted as area integrals of observables weighted by this real probability distribution. Previous studies of the zeros of eigenfunctions of $\mathcal{PT}$-symmetric theories suggest that the zeros interlace and become dense in a narrow band in the complex-$\phi$ plane [18]. Hence, it was conjectured that completeness may have to be defined in terms of area integrals. We suspect a strong connection between the two results and speculate that for $\mathcal{PT}$-symmetric theories a path integral formulation weighted by a real probability can only be achieved in terms of area integrals. That is, we conjecture that a fundamental constraint of $\mathcal{PT}$-symmetric theories is the inability to formulate the path integral as a single integral along a contour in the complex-$\phi$ plane with a weighting function that is pure real. If a pure-real weighting function (taken as $\exp(-S(\phi))$ in conventional field theory) cannot be identified and used as a probability, then Monte Carlo methods are not applicable.

Initially, this seems puzzling in light of recent work on probabilistic interpretations for quantum-mechanical $\mathcal{PT}$-symmetric theories [4]. Bender *et al.* [4] reveal a hidden, extra symmetry in many $\mathcal{PT}$-symmetric Hamiltonians, and this symmetry is denoted by the operator $\mathcal{C}$. Through the use of this symmetry, Bender *et al.* provide a method for calculating explicitly probabilities and demonstrating unitarity. They do this in terms of integrals along contours in the complex-$\phi$ plane for one-dimensional versions of $\mathcal{PT}$-symmetric quantum field theories, *i.e.*, $\mathcal{PT}$-symmetric quantum mechanics. Thus, within quantum mechanics a formulation in terms of area integrals is not necessary, so the interlacing of the zeros of the eigenfunctions, which is also a quantum-mechanical problem, is apparently not relevant to the calculation of probabilities. This suggests that area integrals may also be unnecessary in order for path integrals to be expressed in terms of a weighting function that is a probability. However, the mapping from quantum-mechanical probabilities, which are calculated using the product of complex wave functions, to $\exp(-S(\phi))$, which for Hermitian theories is the weighting function in the path integral and is used as a probability, is non-trivial at best. While certain quantities, such as the ground-state energy, must agree whether calculated using the path integral or quantum-mechanical wave functions, it is not clear that a real formulation of quantum-mechanical probabilities implies that a real formulation of $\exp(-S(\phi))$ or any other weighting function in the path integral even exists. Understanding how this $\mathcal{CPT}$ formalism translates into the path-integral formalism would be a major advance. If the path-integral could be formulated in such a way that its weighting function was real and a probability, then Monte Carlo methods, which are simpler and less computationally exacting than the methods presented here, should be applicable.

We expect that there exists a direct link between the Fokker–Planck probability and the probabilities calculated using the $\mathcal{CPT}$ formalism. It may be that the hidden $\mathcal{C}$ symmetry discovered by Bender *et al.* [4] corresponds in some way to the hidden supersymmetry in the Fokker–Planck equation, but this correspondence is merely speculative and is difficult to intuit. Indeed, based on this work, it seems

likely that this correspondence can only be achieved by means of area integrals, and hence, that the use of Monte Carlo methods and a probabilistic interpretation of $\exp(-S(\phi))$ are not possible. Further study of these Fokker–Planck equations in tandem with the study of the $\mathcal{CPT}$ formalism of Bender *et al.* [4] should allow a fully quantum-field theoretic probabilistic interpretation of $\mathcal{PT}$-symmetric theories, but it remains to be seen whether this will reduce the computational difficulties associated with $\mathcal{PT}$-symmetric theories. In this scenario, a real probabilistic interpretation of $\mathcal{PT}$-symmetric theories would exist, but path integrals would not be useful for calculations. Consequently, the numerical method presented here would still be necessary for calculating physical quantities, and thus, for making predictions. If $\mathcal{PT}$-symmetric theories are ever to be taken seriously as a physical description of the universe and not just as a mathematically interesting problem, such predictions must be made.

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