

### Homework Problem Set 5: Due June 10, 2015

1. If each vessel must be tracked separately and we record the vessel location with a (j,k) scheme and record the radius, length, and linear velocity in double-point precision (8 bytes), then how many vessels can be tracked if we have 8GB of RAM available? Estimate how many levels in the vascular system this corresponds to for an n=2 and for an n=3 bifurcating network?
2. If we are using the Navier-Stokes equations to model our fluid flow through the vascular system, then we need double point precision for both the azimuthal and radial velocity at every “point” within every vessel. If on average we use a 100x100x1000 grid to describe the three-dimensional flow in each vessel (this must be an average because we need a larger grid for describing flow in large versus small vessels and overall an adaptive mesh), then how many vessels can we track? What if it is a 1000x1000x10,000 grid? How does this compare with “lumped” models or models using one-dimensional flow as in problem 1?
3. For an oscillatory pressure gradient defined by  $\Delta p = \Delta p_0 \sin(\omega t)$ , use our inductance formula (ignoring capacitance) to show

$$\dot{Q}_R(t) = \frac{\Delta p_0}{\sqrt{Z^2 + \omega^2 L^2}} \left( \sin(\omega t - \theta) - \frac{\omega L}{\sqrt{Z^2 + \omega^2 L^2}} e^{-t/t_L} \right)$$

where the phase “lag” due to inductance is defined by

$$\theta = \tan^{-1}(\omega t_L)$$

4. a. Show explicitly that when the fluid density is zero, the formula in problem 3 reduces exactly to our Ohm’s Law for fluids, corresponding to no lag to accelerate the fluid.
  - b. Given that the radius of the aorta is approximately 25 mm and the length is 50 mm, calculate the inductance,  $L$ , and impedance,  $Z$ , in the aorta. For the impedance, you should use the value of the Korteweg-Moens velocity calculated in Problem Set 4.
  - c. Using an average heart rate of 60 beats per minute, the frequency,  $\omega$ , can be approximated as 1 Hz. Given this and the results in 4b, calculate the angle  $\theta$  from problem 3. Assuming  $t \gg t_L$ , draw the oscillating pressure gradient and oscillating flow rate, and depict this lag between them.
  - d. Draw similar diagrams that represent the phase lag,  $\theta$ , when it has values of  $0$ ,  $\pi$ , and  $2\pi$ . From these diagrams can you differentiate between the cases for  $0$  and  $2\pi$ . More generally, can you distinguish between these two cases?
5. For an RLC circuit with all elements in series, the differential equation is

$$L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = V(t)$$

- a. Fourier transform this equation to obtain

$$\tilde{q} = \frac{\tilde{V}}{L\left(\frac{1}{t_L t_C} - \omega^2 + i\frac{\omega}{t_L}\right)} = \frac{\tilde{V}}{L\left(\omega_0^2 - \omega^2 + i\frac{\omega}{t_L}\right)}$$

where  $\omega_0^2 = 1/t_L t_C$ .

- b. Multiply this by its complex conjugate to find

$$|\tilde{q}|^2 = \frac{|\tilde{V}|^2}{L^2\left(\left(\omega_0^2 - \omega^2\right)^2 + \left(\frac{\omega}{t_L}\right)^2\right)}$$

- c. Find the roots of the denominator and express it in terms of  $1/t_L$  and  $t_L/t_C$ . Does this denominator have any real roots? Plot by hand the function in 6b versus  $\omega$  for the case where there is and is not a real root of  $\omega$ . Discuss the ideas of resonance and what this means for the ratio  $t_L/t_C$  and our circuit.

### EXTRA CREDIT

6. Impedance is often assumed to not vary in time. Here, we will investigate how this might be dealt with.

- a. Using our expression for the impedance in the acoustic limit,  $Z = \frac{\rho c_0}{\pi r^2}$ , express  $dZ/dt$  in terms of  $Z$  and  $dr/dt$ .
- b. Starting with the definition of the capacitance,  $C$ , and assuming that vessel length is changing with time, show that

$$\frac{dr}{dt} = \frac{C}{2\pi r l} \frac{d(\Delta p)}{dt}$$

and using this relationship, show that

$$\frac{dZ}{dt} = \frac{-t_c}{V} \frac{d(\Delta p)}{dt}$$

where  $t_c$  is the time constant from capacitance,  $ZC$ , as defined in class, and  $V$  is the volume of the vessel.

- c. Since the volume flow rate and impedance are now both changing with time, we cannot define a characteristic time constant, but we can define a characteristic volume flow rate based on the above equation. What is it?