Problem Set 4 for Biomath 213: Due May 29, 2015

1. Calculate the Korteweg-Moens velocity for the canine cardiovascular system, given that the canine aorta has a vessel wall thickness of 4 mm, a radius of 25 mm, vessel wall elasticity of 4.8×10^5 kg/(ms^2). How does this compare to the speed of sound?
2. a. Given a vessel at level k with a fixed volume flow rate, if the power expended to deal with dissipation and reflections are the same, what does this imply about the values of the impedances? Why?
   b. By setting the dissipation and reflection impedances equal to one another, derive a formula for the ratio of the vessel radius squared to vessel length, r^2/l, in terms of the viscosity, density, and Korteweg-Moen's velocity.
   c. Using the scaling relationships for r and l in the dissipation regime, derive the number of levels from the capillaries to the level at which the transition from reflections to dissipation occurs, $N_{transition}=N-k_{transition}$.
   d. Calculate $N_{transition}$ and $r_{transition}$ using the values from problem 1 and a viscosity of 0.04 Ns/m^2, a capillary radius of 4 μm, a capillary length of 80 μm, and a branching ratio of n=2.
3. a. In our expansions of the Bessel functions, we take the limit of kR<<1 or kR>>1. At the intermediate point, |k|R=1, what value of the vessel radius R does this correspond to? Use a wave frequency for humans equal to the heart rate (i.e., the frequency of the pressure pump), which is approximately 60 beats per minute or 1 Hz. How does this value compare to the answer in 2d?
   b. If heart rate scales like body mass to the negative ¼ power (i.e., $M^{-1/4}$), then what does this imply about the heart rate of a shrew with a body mass of 1g relative to a human with a body mass of approximately 100 kg? Using this information, what value of R corresponds to |k|R=1 for a shrew? How does this answer compare to 2d?
   c. In terms of mass dependence how do the results in problem 3 differ from those in problem 2?
4. How do the radius of the capillary and the radius of an aorta in a human compare with the values calculated in problems 2 and 3? What regimes (dissipation or reflection) are these in respectively?
5. In this problem we will show

$$\left(\frac{c}{c_0}\right)^2 \sim -\frac{J_2(ix_0)}{J_0(ix_0)}$$

   a. Using the massless approximation ($\rho_w=\sigma=0$), substitute Eqs. 3a and 4a into Eqs. 7a and 8a and take the ratio of the resultant equations to get

$$\frac{k}{k'}\left(\frac{J_1(ik'R)}{J_0(ik'R)-\frac{\mu\omega}{Ehk^2}J_1(ik'R)\frac{(k'^2+k^2)}{k'}}\right)=\frac{J_1(ikR)-\frac{i\rho\omega^2R^2}{Ehk}J_0(ikR)}{J_0(ikR)+\frac{2\mu\omega}{Ehk}J_1(ikR)}$$

   b. Put this equation in dimensionless form by using the identities x=kR, x'=k'R,

$x_0=k_0R$, $k_0^2=i\omega\rho/\mu$, and $y=c/c_0$.

$$\frac{x}{x'}\left(\frac{J_1(ix')}{J_0(ix')-\frac{iy^2}{2x_0^2}J_1(ix')\frac{(x'^2+x^2)}{x'}}\right)=\frac{J_1(ix)-\frac{1}{2}ixy^2J_0(ix)}{J_0(ix)+\frac{ixy^2}{x_0^2}J_1(ix)}$$

c. Using the approximation $x<<1$ along with Bessel expansions and $x'\sim x_0$, show that

$$1-y^2=\frac{\frac{2}{ix_0}J_1(ix_0)}{J_0(ix_0)+\frac{1}{4}y^2\left(\frac{2}{ix_0}J_1(ix_0)\right)}$$

d. Realizing that c_0 is like an upper speed limit, we see that $y<<1$, so that we can ignore the second term in the denominator. Using the recursion relation for Bessel functions

$$\left(\frac{c}{c_0}\right)^2=y^2\sim-\frac{J_2(ix_0)}{J_0(ix_0)}$$

6. Consider a vascular network that is grid-like, such that nearest neighbors are all connected and each branching junction has three vessels connected to it (like bifurcating branching (n=2) for a hierarchical tree). Starting from the heart (i.e., some central location), the blood can go through a vessel either to the "right" or "left". When the blood reaches the next branching junction (i.e., node), it cannot travel back through the vessel from which it entered, and it must go the opposite direction (right versus left or vice versa) than it did at the previous node due to a series of inter-connected valves. As the blood continues to traverse the network, it must alternate directions through vessels forever after that. Prove that if this pattern continues (including in the venous system if bloods enter there) that the blood must eventually return to the heart. This is equivalent to proving that the flow cannot get stuck traveling forever through a loop of vessels of which the heart is not a part. (Hint: Consider the "dual" network constructed by replacing each vessel with a node at its mid-point and connecting these new nodes only if the original vessels were connected at an original node. Map the flow through the original network onto what flow means through the new "dual" network and consider what this means about loops. In the figure below, the first diagram is a path through the original network, and the second diagram is a path through the dual network.)