**Problem Set 3 for Biomath 213: Due May 13, 2015**

1. For a Newtonian fluid at steady state with laminar flow only in the z direction, solve for the velocity $v_z$ in the following cases by including the effect of gravity, $|\rho\nabla\phi| = |\rho g|$, where g is the acceleration due to gravity.

a. A pipe with long axis vertical to the ground and fluid flowing down towards the earth.

b. A pipe with long axis vertical to the ground and fluid flowing up away from the earth. (Hint: You should be able to do parts a and b by adapting an equation derived in class, and not solving from scratch.)

2. Discuss how solutions for velocities will be affected for a fluid flowing through a pipe with long axis horizontal to the ground for a vessel with elastic walls. You do not have to give the solution in this case, but write down the Navier-Stokes equations and explain how this differs from problem 1.

3. Thus far, we have ignored any dependence on the angle $\theta$ by assuming rotational symmetry. This problem is to give some intuition for when this is not the case.

a. Just looking at the case of laminar flow, if we assume that $v_z$ depends on $\theta$ and can be expressed as u(r,$\theta$,t), are we able to use separation of variables in the Navier-Stokes equations. Why or why not?

b. Notice that as long as the equations $\frac{\partial^2 v_z}{\partial\theta^2} = 0 = \frac{\partial^2 v_r}{\partial\theta^2}$ hold then our Navier-Stokes equations are unchanged. Therefore, if $\Theta(\theta)$ is an arbitrary function of $\theta$, solutions to the general equation

$$\frac{\partial^2\Theta(\theta)}{\partial\theta^2} = 0$$

can multiply $v_z$ and $v_r$ such that these new velocity components are still solutions to the Navier-Stokes equation. What is the solution of $\Theta(\theta)$ for this general equation?

c. If we think of rotational symmetry as a boundary condition, how does this constrain $\Theta(\theta)$? For this case, do $v_z$ and $v_r$ change from our solutions in class? What happens if instead we apply the boundary conditions that $\Theta(0) = 0$ and that $\Theta(\pi)$ is maximal (i.e., $\left.\frac{\partial\Theta}{\partial\theta}\right|_{\theta=\pi} = 0$), similar to the radial boundary conditions?

4. a. The function $\Theta(\theta) = \pi - |\pi - \theta|$ satisfies the latter boundary conditions in part c: $\Theta(0) = 0$ and $\Theta(\pi)$ is maximal (i.e., $\left.\frac{\partial\Theta}{\partial\theta}\right|_{\theta=\pi} = 0$). Give a physical interpretation of what this function means for fluid flow and discuss what it means for turbulence? Is

this function a valid solution for $\Theta(\theta)$ such that it satisfies the general equation in problem 3b? If this function multiples $v_z$ and $v_r$, are those still valid solutions to the Navier-Stokes equations? Why did we not find this solution in problem 3c?

b. If instead we have $\frac{\partial^2 v_z}{\partial\theta^2} = -c^2 = \frac{\partial^2 v_r}{\partial\theta^2}$ where c is a constant, and the equation

$$\frac{1}{\Theta(\theta)}\frac{\partial^2\Theta(\theta)}{\partial\theta^2} = -c^2$$

how would this affect our solutions for $v_z$ and $v_r$? More specifically, how would this affect the Bessel equations we obtain for the radial component? Now, what is the solution of $\Theta(\theta)$ for this general equation? What happens if we impose the boundary conditions: $\Theta(0)=0$ and $\Theta(\pi)$ is maximal (i.e., $\left.\frac{\partial\Theta}{\partial\theta}\right|_{\theta=\pi} = 0$)?

5. a. Using the formula for derivatives of Bessel functions given in problem 6 below, and the fact that $J_{-1}(x) = -J_1(x)$, show that

$$\frac{dJ_0(x)}{dx} = -J_1(x)$$

b. Using the recursion relation and the formula for derivatives in part problem 7, show that

$$\frac{d(xJ_1(x))}{dx} = xJ_0(x)$$

so that

$$\int dx x J_0(x) = xJ_1(x)$$

**EXTRA CREDIT**

6. To learn about Bessel functions, start with the recursion relation

$$J_n(x) = \frac{x}{2n}(J_{n-1}(x) + J_{n+1}(x))$$

and substitute this into the differential equation for $J_n(x)$ and show that by using the Bessel equations for $J_{n-1}(x)$ and $J_{n+1}(x)$ and repeated use of the recursion relation, you can derive the formula for derivatives

$$2\frac{dJ_n(x)}{dx} = J_{n-1}(x) - J_{n+1}(x)$$