Problem Set 2 for Biomath 213: Due May 1, 2015

1. a. Calculate the Reynolds number in a human aorta with diameter 20 mm and average linear velocity 8.5 cm/s. What does this indicate about laminar versus turbulent flow in the cardiovascular system? Will the Reynold's number increase or decrease as blood travels through the cardiovascular system down towards the capillaries?
   b. Discuss the approximation of blood as a continuous fluid in a human aorta with diameter 20 mm versus the approximation in a capillary with diameter 10 μm given that red bloods cells are 10 μm in diameter.

2. The purpose of this problem is to derive the Navier-Stokes equations in cylindrical coordinates (r,θ,z) by performing a change of variables from the Navier-Stokes equations in Cartesian coordinates (x,y,z). The z (axial) coordinate remains unchanged, and the (x,y) coordinates are mapped onto a circle of radius r such that $x=r\cos(\theta)$, $y=r\sin(\theta)$, and $x^2+y^2=r^2$.

   a. Prove the relationships: $r\frac{d}{dr}=x\frac{d}{dx}+y\frac{d}{dy}$ (sum of components)

   $$\frac{d}{d\theta}=-y\frac{d}{dx}+x\frac{d}{dy} \text{ (mixing of components)}$$

   b. Using the definitions for velocity in cylindrical coordinates—v_r=dr/dt, v_r=r dθ/dt, and v_z=dz/dt—derive the relationships:

   $$rv_r = xv_x + yv_y \quad \text{(sum of components)}$$

   $$rv_\theta = -yv_x + xv_y \text{ (mixing of components)}$$

   c. Use the relationships from parts a and b to show that

   $$\nabla \bullet \vec{v} = \frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}=\frac{1}{r}\frac{\partial (rv_r)}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial v_z}{\partial z}$$

   and thus identify the gradient

   $$\nabla = \frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}=\frac{1}{r}\frac{\partial (r-)}{\partial r}+\frac{1}{r}\frac{\partial}{\partial \theta}+\frac{\partial}{\partial z}$$

   where the – is a place holder.

   d. Use the relationships from parts a and b to derive the Laplacian

   $$\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^2}{\partial z^2}$$

   Can you get this result by twice applying the gradient in part c?

   e. Use the relationships from parts a and b to show that

   $$\vec{v} \bullet \nabla = v_x\frac{\partial}{\partial x}+v_y\frac{\partial}{\partial y}+v_z\frac{\partial}{\partial z}=v_r\frac{\partial}{\partial r}+\frac{1}{r}v_\theta\frac{\partial}{\partial \theta}+v_z\frac{\partial}{\partial z}$$

f. Using results from parts c-e, it should now be straightforward to obtain the Navier-Stokes equations for the z component

$$\rho\left(\frac{d}{dt}+v_r\frac{\partial}{\partial r}+\frac{1}{r}v_\theta\frac{\partial}{\partial\theta}+v_z\frac{\partial}{\partial z}\right)v_z=\frac{-\partial p}{\partial z}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}\right)v_z$$

Why is this component relatively easy?

g. Derive expressions for $(\vec{v}\bullet\nabla)v_x$ and $(\vec{v}\bullet\nabla)v_y$ in cylindrical coordinates. What value of θ corresponds to the correct equations for the r and θ components of the Navier-Stokes equations? More generally, by matching $\sin(\theta)$ and $\cos(\theta)$ terms (you do not have to do this), you can derive the equations

$$\rho\left(\frac{d}{dt}+v_r\frac{\partial}{\partial r}+\frac{v_\theta}{r}\frac{\partial}{\partial\theta}+v_z\frac{\partial}{\partial z}-\frac{v_\theta^2}{r}\right)v_r=-\frac{\partial p}{\partial r}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}+\frac{\partial^2 v_r}{\partial z^2}-\frac{v_r}{r^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}\right)$$

$$\rho\left(\frac{dv_\theta}{dt}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta}+v_z\frac{\partial v_\theta}{\partial z}-\frac{v_\theta v_r}{r}\right)=-\frac{1}{r}\frac{\partial p}{\partial\theta}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2}+\frac{\partial^2 v_\theta}{\partial z^2}-\frac{v_\theta}{r^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}\right)$$

3. a. Defining the vorticity as $\Omega=\nabla\times\vec{v}$ and ignoring the viscosity and shear forces (i.e., $\mu\nabla^2\vec{v}$ =0), rewrite the vector form of the Navier-Stokes equations as

$$\frac{d\vec{v}}{dt}+\Omega\times\vec{v}+\frac{1}{2}\nabla v^2=-\frac{\nabla p}{\rho}-\nabla U$$

where U is the potential energy per unit mass of an external force.

b. Show that

$$\frac{d\Omega}{dt}+\nabla\times(\Omega\times\vec{v})=0$$

c. If Ω=0 initially, can it change in time? What if we add viscosity?

4. a. Use the equation in 3a to show that when velocity is constant in time (but not space), Bernoulli's theorem-- $\frac{1}{2}v^2+\frac{p}{\rho}+U=const$ --comes out of the equations. This is just a statement of conservation of energy.

b. Use Bernoulli's theorem to derive the velocity of flow out of a heart with a hole at the bottom side, given that the hole is sufficiently small and the heart sufficiently large that the height of the water is not noticeably changing.

5. The Womersley's number, Ω, is a dimensionless number that captures differences in velocity for smooth flow versus pulsatile flow. These differences

are a function of the fluid density, ρ, the vessel radius, *r*, the frequency of oscillation, ω, and the viscosity, μ. Since it is a dimensionless ratio, we are free to choose the power of the exponent of the ratio, as we did in class for the Reynold's Number, and we do this by choosing Ω to depend linearly on *r.* Using dimensional analysis, derive the Womersley's number.

6. Using our results from class for general solutions of the velocity, *v*(*r*,*t*), in a rigid tube with a time dependent pressure gradient, solve for the velocity when $\tilde{p}_0(t) = \tilde{p}_0\theta(t) = (-p_0/l)\theta(t)$ where $\tilde{p}_0$ is a constant and θ(t) is the Heaviside or theta function, which is 0 for times *t*<0 and is 1 for times *t*>0. Express your answer in terms of the poles of the integral as commonly used to perform residue integrals in complex analysis.