Problem Set 1 for Biomath 213: Due April 20, 2015

1. Given the following data for a human pulmonary vascular system.

Level	Number of units	Vessel diameter (mm)	Vessel length (mm)
0	1.00E+00	20.78±1.81	20.12±9.64
1	2.00E+00	7.31±1.38	9.39±5.37
2	4.00E+00	4.33±0.68	5.01±3.18
3	1.00E+01	2.81±0.46	3.59±2.67
4	2.40E+01	1.78±0.25	2.16±1.38
5	4.10E+01	1.17±0.14	1.54±1.19
6	9.20E+01	0.77±0.1	1±0.66
7	1.64E+02	0.51±0.06	0.76±0.51
8	3.16E+02	0.35±0.09	0.64±0.33
9	6.47E+02	0.22±0.03	0.54±0.32
10	1.31E+03	0.15±0.02	0.48±0.24
11	2.78E+03	0.096±0.015	0.31±0.15
12	8.71E+03	0.056±0.005	0.26±0.16
13	2.52E+04	0.036±0.006	0.19±0.1
14	7.63E+04	0.02±0.003	0.20±0.08
15	2.27E+05		
16	6.84E+05		

   a. What is the average branching ratio, *n*, and what are the scaling ratios and average scaling exponents, *a* and *b*, for the radii, $\beta = r_{k+1}/r_k$, and length, $\gamma = l_{k+1}/l_k$, expressed in terms of the branching ratio? Assume a symmetrically branching network and round your answers where appropriate.
   b. Are there any indications that the branching ratio may be changing throughout the network or that there may be asymmetric branching in the network? Explain.

2. Assuming roughly 8,000,000,000 capillaries in a human vascular system:

a. For a bifurcating tree (n=2), how many levels does this suggest are in the human cardiovascular system to traverse from the heart to the capillaries?
b. If the radius of the heart is r_0=25mm, and the radius of the capillaries is r_N=8μm, what is the ratio of the linear velocity in the heart relative to that in the capillaries?

3. a. Write down expressions for how the radius and length of a vessel at an arbitrary level k, r_k and l_k, relate to the radius and length of vessels at level k=8.
   b. Calculate the total network volume, V_{net}, relative to the dimensions at the level k=8. Do the same relative to an arbitrary but fixed level, k'.

4. Using the scaling relationships $\beta = r_{k+1}/r_k = n^{-1/2}$ and $\gamma = l_{k+1}/l_k = n^{-1/3}$:
   a. Calculate the total path length, l_{TOT}, from the heart to the capillary expressed in terms of the length of a capillary l_N. If l_N=16μm, then what is your estimate for the total path length for vascular system with N=25 levels?
   b. What is the leading-order scaling exponent for total path length as related to the number of capillaries, N_N, when the number of levels N->Infinity?
   c. Calculate dln(l_{TOT})/dln(N_N) for the exact equation from the geometric series to get the second-order effective exponent for the relationship between total path length and the number of capillaries.
   d. Calculate an estimate for the total transit time for blood to travel from the heart to the capillaries in terms of the length of the capillary, l_N, and the linear velocity in the capillary, u_N. If l_N=16μm, u_N=0.6 mm/s, and the branching ratio n=2, what is your estimate for the total transit time? How would this change if $\beta = r_{k+1}/r_k = n^{-1/3}$.

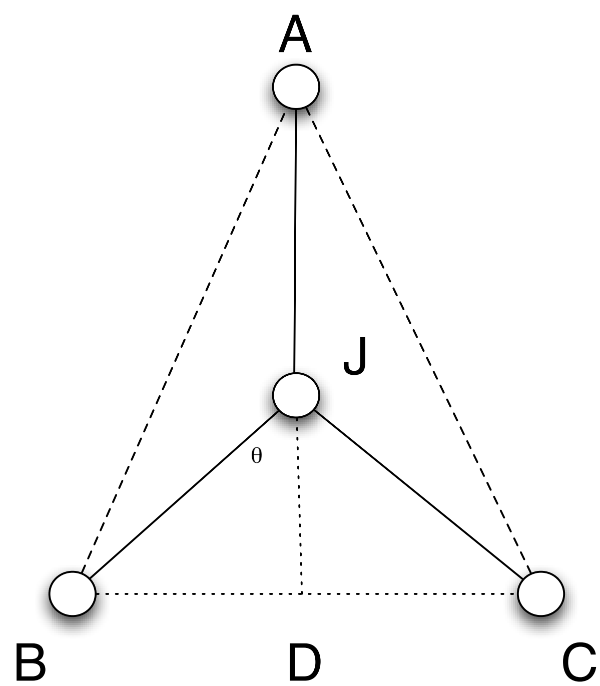
5. Using our equations for whole network volume, $V_{net} = V_N \frac{N_N^{(2a+b)} - n^{1-2a-b} N_N}{1 - n^{1-2a-b}}$, investigate the limit as the number of levels N-> Infinity for the following three cases. Find the asymptotic solutions and interpret what they mean.
   a. 2a+b>1,
   b. 2a+b<1,
   c. 2a+b=1

6. For an asymmetric network with bifurcating branching and a fixed number of levels from the heart to the capillaries, calculate:
   a. The average area of a capillary in terms of the radius of the aorta, r_0, the number of levels, N, the branching ratio, n, and the asymmetric ratios $\beta_B = r_{B,k+1}/r_k$ and $\beta_s = r_{s,k+1}/r_k$. What is the result for area preserving branching? What happens in the limit the number of levels N->Infinity?
   b. The variance for the area of a capillary in terms of the radius of the aorta, r_0, the number of levels, N, the branching ratio, n, and the asymmetric ratios $\beta_B = r_{B,k+1}/r_k$ and $\beta_s = r_{s,k+1}/r_k$. Can you give a result for area-

preserving branching? Why physically is this result less specific than for the average? What happens in the limit that the number of levels *N*->Infinity?

c. How are the N->Infinity results affected if the cross-sectional area of the aorta scales linearly with the number of terminal units, N_N? That is, as we make the network larger (i.e., make the organism larger), the aorta increases systematically with it.

7. Using our model for the optimal angle, $\theta_{opt}=\pi/3$, that minimizes construction material, calculate how long the length of the pre-branch segment, $l_{A->J}$, is relative to the length of the full segment, $l_{A->D}$, that bisects the line between B->C in the limits:
a. $l_{A->D} >> l_{B->C}$, the length of the bisecting line is much greater than the distance from B to C
b. $l_{A->D} << l_{B->C}$, the length of the bisecting line is much less than the distance from B to C. What goes wrong in this limit? Interpret it geometrically.



8. Use dimensional analysis to determine an equation for the velocity of a wave of blood through a single vessel. Use intuition to determine which of the parameters from class this velocity should depend on. If possible, derive an equation with the correct power-law dependence on this combination of parameters. If the combination of some parameters cannot be uniquely determined from dimensional analysis, explain why. If you know the equation for the speed of sound or speed of a wave along a string, that may be helpful here.