General solution for time-dependent pressure gradient

Combining all of this back together

$$u(r,t) = \Psi(t) - \int_{-\infty}^{\infty} d\omega e^{i\omega t} \tilde{\Psi}(\omega) \frac{J_0(kr)}{J_0(kR)}$$

where

$$\Psi(t) = -\frac{1}{\rho}\int_0^t \hat{p}(t)\,dt$$

and

$$k^2 = \frac{\omega\rho}{i\mu}$$

$$\tilde{\Psi}(\omega) = \int_{-\infty}^{\infty} \frac{dt}{2\pi} \Psi(t) e^{-i\omega t}$$

Check that constant pressure gradient reduces to same formula as before

a. Check we get correct result for constant pressure gradient

$$\hat{p}(t) = \frac{dp}{dz} = \frac{p(l) - p(0)}{l - 0} = -\frac{p}{l}$$

For constant pressure gradient must use Dirac delta functions

$$\int_{-\infty}^{\infty} dx\, f(x)\delta(x - x_0) = f(x_0)$$

Like a distribution that is infinite at a single point, x_0, and saturates the integrand there.
A common identity in Fourier space is

$$\delta(\omega - \omega_0) = \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{i(\omega-\omega_0)t}$$

At w=w_0 this gives Infinity, and when w does not equal w_0, this gives zero because the oscillations cancel out over every 2p interval all the way to Infinity.
Can also think of delta function as a distribution like a Gaussian in the limit that the width of the distribution goes to 0 and the height of the peak goes to Infinity.

b. Oscillatory pressure gradient

$$\frac{dp(z,t)}{dz} = \hat{p}(t) = \hat{p}_0 e^{i\omega_0 t} = -\frac{p_0}{l} e^{i\omega_0 t}$$

$$u(r,t) = e^{i\omega_0 t} \frac{i\hat{p}_0}{\rho\omega_0} \left[1 - \frac{J_0(k_0 r)}{J_0(k_0 R)} \right]$$

where

$$k_0^2 = \frac{\omega_0 \rho}{i\mu}$$

Calculate flow rate

For oscillatory pressure gradient

$$\dot{Q} = \frac{\pi p_0 R^4}{8\mu l}\left[\frac{8J_2(k_0R)}{(k_0R)^2 J_0(k_0R)}e^{i\omega_0 t}\right]$$

↑

Poiseuille flow
Newtonian fluid

Starting from Bessel function differential equation, all kinds of tricks can be played to find recursion relations, derivatives, and integrals. You will have a homework problem that guides you through this.

Calculate impedance

For oscillatory pressure gradient

$$Z = \frac{p}{\dot{Q}} = \frac{8\mu l}{\pi R^4}\left[\frac{(k_0 R)^2 J_0(k_0 R)}{8 J_2(k_0 R)}\right]$$

↑

Poiseuille flow

Captures energy loss due to dissipation and reflections

Power dissipated

Our flow rate and other properties are now complex, so we have an equations for each Real and imaginary part, so now we have 2*3=6 separate equations. Take the dot product of the real part of the velocity, v, and integrate over the volume for both sides of the Navier-Stokes equation

$$\frac{dE}{dt} = \int d^3x(\mathrm{Re}[\vec{v}] \bullet \mu\nabla^2 \mathrm{Re}[\vec{v}])$$

The time average over oscillations can be expressed as

$$\left\langle \frac{dE}{dt} \right\rangle = \frac{1}{2}\mu \mathrm{Re}\left[\int d^3x(u^*(r,t) \bullet \nabla^2 u(r,t))\right] = -\pi\mu l \int_0^R drr\left|\frac{\partial u}{\partial r}\right|^2$$

$$\left\langle \frac{dE}{dt} \right\rangle = -\frac{\pi p_0^2}{\omega_0 \rho l}\int_0^R drr\left|\frac{J_1(k_0 r)}{J_0(k_0 R)}\right|^2$$

Power dissipated

In the limit of very slow oscillations and almost steady flow, w_0<<1 and k_0<<1

$$\frac{J_1(k_0 r)}{J_0(k_0 R)} \sim \frac{\frac{1}{2} k_0 r}{1} \sim \frac{1}{2} k_0 r$$

So for slow/weak oscillations, we have

$$\left\langle \frac{dE}{dt} \right\rangle = \frac{\pi p_0^2}{16 \mu l} R^4 = \frac{1}{2}\left(\frac{dE}{dt}\right)_{Newtonian}$$

Half the energy is dissipated but half goes to the kinetic energy of keeping the fluid moving back and forth (i.e., oscillating).