Modeling Vascular Networks with Applications

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Class project

1. 10 page paper

2. 15 minute presentation + 5 minutes questions

3. Based on published ideas in primary journal article in scientific literature

4. Summary of published work

5. Extension of previous work either through analytical modeling, computation, analysis of new data, or some other advance

6. Good to pick your own topic but get my approval first. If you can't come up with topic, I can help you find one.

General solution for rigid tube with time-dependent pressure gradient

Time-dependent pressure gradient

$$\frac{dp(z,t)}{dz} = \hat{p}(t)$$

$$p(z,t) = \hat{p}(t)z + p_0(t)$$

Navier-Stokes equation becomes

$$\rho \frac{du(r,t)}{dt} + \hat{p}(t) = \mu \frac{1}{r} \frac{d}{dr}\left(r \frac{du(r,t)}{dr} \right)$$

Making the ansatz that we can write the solution in the form

$$u(r,t) = \overline{u}(r,t) + \Psi(t)$$

And we can choose to separate this into two equations

$$\rho \frac{d\Psi(t)}{dt} = -\hat{p}(t)$$

and

$$\rho \frac{d\overline{u}(r,t)}{dt} = \mu \frac{1}{r} \frac{d}{dr}\left(r \frac{d\overline{u}(r,t)}{dr} \right)$$

Solving the first equation

$$\rho \frac{d\Psi(t)}{dt} = -\hat{p}(t)$$

Integrating this yields

$$\Psi(t) = -\frac{1}{\rho}\int_0^t dt\,\hat{p}(t)$$

To solve the second equation we use separation of variables

$$\bar{u}(r,t) = R(r)T(t)$$

We can re-arrange the equation to obtain

$$\frac{1}{T(t)}\frac{dT(t)}{dt} = \frac{\mu}{\rho}\frac{1}{R(r)}\frac{1}{r}\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right)$$

Each side of this equation varies independently, so each must separately be equal to a constant. To solve this we now need to know a little bit about Fourier transforms and Bessel functions.

Fourier transforms

$$\Im[T(t)] = \int_{-\infty}^{\infty} \frac{dt}{2\pi} T(t) e^{-i\omega t} = \tilde{T}(\omega)$$

$$T(t) = \int_{-\infty}^{\infty} d\omega \tilde{T}(\omega) e^{i\omega t} = \Im^{-1}[\tilde{T}(\omega)]$$

Taking the Fourier transform of the derivative of T(t) and using integration by parts along with the condition that T(t) vanishes at +/- Infinity (a BC) gives

$$\Im\left[\frac{dT(t)}{dt}\right] = i\omega \Im[T(t)] = i\omega \tilde{T}(\omega)$$

Fourier transforms

$$\Im\left[\frac{d^n T(t)}{dt^n}\right] = i\omega\Im\left[\frac{d^{n-1} T(t)}{dt^{n-1}}\right] = \ldots = (i\omega)^n \tilde{T}(\omega)$$

An nth order derivative becomes an nth-order exponent, so that an nth order differential equation becomes an nth order polynomial equation. Solve algebraic equation and in then invert to get solution in original space. We will use these a lot.

Can transform in one variable, such as time, or multiple variables, such as time and space. Just be careful and make sure we apply it consistently.

Fourier transforms

$$e^{i\omega t} = \cos\omega t + i\sin\omega t$$

Can derive this directly from series representations. Shows that this transform is intimately connected to sine and cosine functions, and since we are integrating over ω, we are looking at all possible combinations. Maybe we'll talk later about how this can be used to decompose oscillatory functions into sine and cosine components.

You can do a huge amount of complex analysis just understanding this function, and we will be doing that.

Our other piece of the separation of variables equation then becomes

$$\frac{d^2R(r)}{dr^2} + \frac{1}{r}\frac{dR(r)}{dr} - \frac{i\omega\rho}{\mu}R(r) = 0$$

We can change variables with r'=kr, so that the solution is obvious as a zeroth-order Bessel function

$$R(r) = J_0(kr)$$

where

$$k^2 = \frac{-i\omega\rho}{\mu}$$

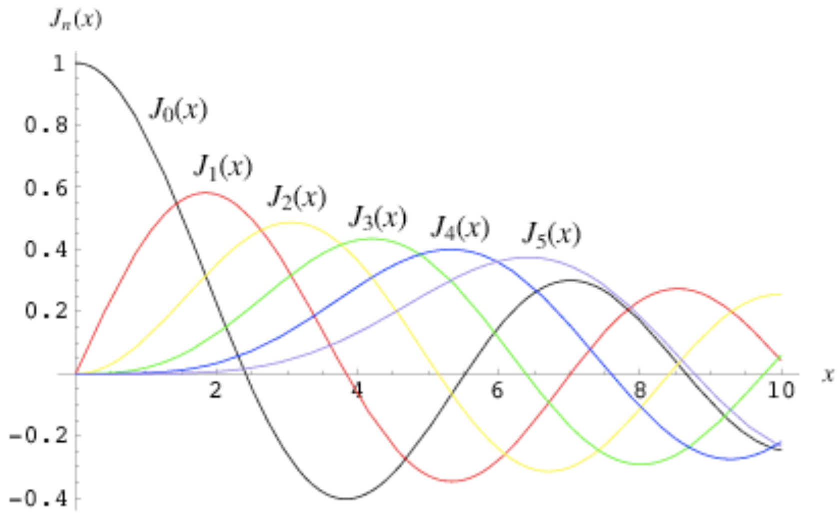
Bessel equations

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - \nu^2) y = 0$$

Equations appear often for problems with cylindrical or spherical symmetry.

$$J_\upsilon(x)$$

Like sine function or other trigonometric functions on a triangle or circle.



Putting our solution back together and remembering that we are still in Fourier space

$$\Im[\overline{u}(r,t)] = \tilde{T}(\omega)J_0(kr)$$

Inversing the Fourier transform and thus integrating from frequency space back to time space

$$\overline{u}(r,t) = \Im^{-1}[\tilde{T}(\omega)J_0(kr)] = \int_{-\infty}^{\infty} d\omega e^{i\omega t}\tilde{T}(\omega)J_0(kr)$$

Putting all our solutions together we have

$$u(r,t) = \Psi(t) - \int_{-\infty}^{\infty} d\omega e^{i\omega t} \tilde{T}(\omega) J_0(kr)$$

Applying the boundary condition at the tube wall, u(R,t)=0, and again inverting the Fourier transform, we find

$$\tilde{T}(\omega) = -\frac{\tilde{\Psi}(\omega)}{J_0(kR)}$$

General solution for time-dependent pressure gradient

Combining all of this back together

$$u(r,t) = \Psi(t) - \int_{-\infty}^{\infty} d\omega e^{i\omega t} \tilde{\Psi}(\omega) \frac{J_0(kr)}{J_0(kR)}$$

where

$$\Psi(t) = -\frac{1}{\rho}\int_0^t \hat{p}(t)\,dt$$

and

$$k^2 = \frac{\omega\rho}{i\mu}$$

$$\tilde{\Psi}(\omega) = \int_{-\infty}^{\infty} \frac{dt}{2\pi} \Psi(t) e^{-i\omega t}$$

Check that constant pressure gradient reduces to same formula as before