# Modeling Vascular Networks with Applications

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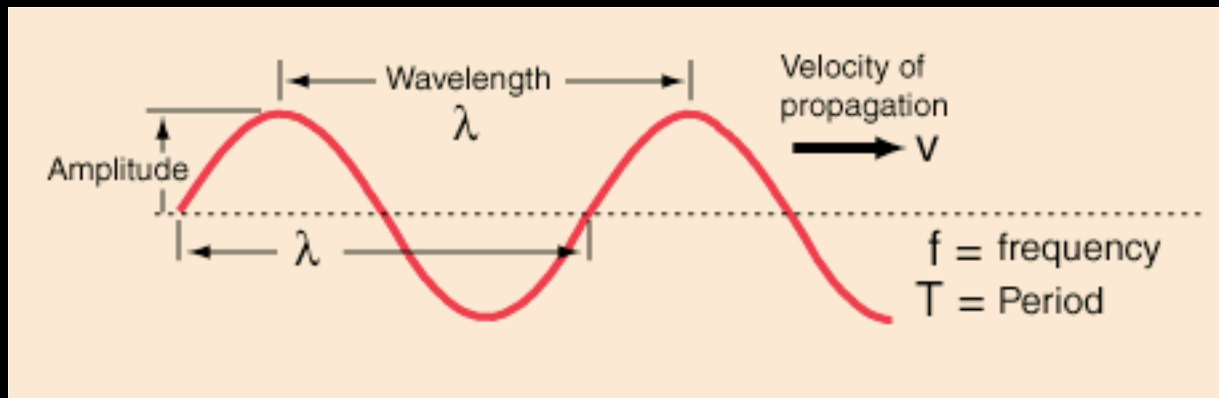
Spring 2015 Quarter

# Wave motion

# Wave velocity




wave frequency　　　wave number

$$c = \frac{\omega}{k} = \frac{2\pi f}{(2\pi / \lambda)} = f\lambda$$

wave (phase) velocity

# Our wave is forward traveling

$$e^{i(\omega t - kz)} = e^{ik(ct - z)}$$

If you move a marker along with velocity z=ct, then you get $e^0$, which is 1, so no change. Thus, any point on the wave is moving forward with velocity ct.

$$e^{i(\omega t + kz)} = e^{ik(ct + z)}$$

This equation thus represents a backward moving wave that we would see if a wave is reflected.

Time and space are interchangeable
http://www.acs.psu.edu/drussell/Demos/wave-x-t/wave-x-t.html

# Group velocity

For waves with multiple frequency components, there is also a speed with which packets of various amplitude travel

$$c = \frac{d\omega}{dk}$$

wave (group) velocity

These packets of the wave are what carry actual information because they have actual structure, so this is especially important for applications like neuroscience.

http://en.wikipedia.org/wiki/Group_velocity

# Two types of waves often described

Transverse waves—oscillations are perpendicular to motion of wave (what we've been looking at)

Longitudinal waves—oscillations in same direction as motion of wave

Water waves and waves of blood and along vessel walls are a **mix** of these of these two types of waves

http://www.acs.psu.edu/drussell/demos/waves/wavemotion.html

# Korteweg-Moens velocity

# Derive Korteweg-Moens velocity

This is the velocity of the wave if viscosity is set to zero. So, it's how fast the wave could travel if no viscosity was slowing it down. Therefore, we can set the homogeneous part of the velocity to 0. Then, the ratio of Eqs. 7a and 8a demand

$$\frac{\tilde{\xi}_r^I(k)}{\tilde{\xi}_z^I(k)} = \frac{J_1(ikR)}{J_0(ikR)}$$

Eq. 8a by itself implies

$$i\omega\tilde{\xi}_z^I(k) = \frac{k}{\omega\rho}\tilde{p}(k)J_0(ikR)$$

Eq. 3a by itself implies for the massless approximation ($r_w$=s=0) is

$$\frac{hE}{R^2}\tilde{\xi}_r^I(k) = \tilde{p}(k)J_0(ikR) = \frac{i\omega^2\rho}{k}\tilde{\xi}_z^I(k)$$

# Boundary conditions (Eqs. 7 and 8) give

$$i\omega\tilde{\xi}_r(k,\omega) = \frac{k}{k'}\tilde{v}_z^H(k,\omega)J_1(ik'R) + \frac{k}{\omega\rho}\tilde{p}(k,\omega)J_1(ikR) \quad \text{(7a)}$$

$$i\omega\tilde{\xi}_z(k,\omega) = \tilde{v}_z^H(k,\omega)J_0(ik'R) + \frac{k}{\omega\rho}\tilde{p}(k,\omega)J_0(ikR) \quad \text{(8a)}$$

This now gives us 4 equations and 4 unknown arbitrary functions, so we have everything we need.

# Equations of motion for wall become

$$h\tilde{\xi}_r(k,\omega)\left[\frac{E}{R^2}-\omega^2\rho_w\right]=\tilde{p}(k,\omega)J_0(ikR)+ik\frac{Eh\sigma}{R}\tilde{\xi}_z(k,\omega) \quad (3a)$$

$$h\tilde{\xi}_z(k,\omega)\left[\omega^2\rho_w-k^2E\right]=\frac{iEh\sigma}{R}k\tilde{\xi}_r(k,\omega)+i\mu\left[\tilde{v}_z^H(k,\omega)\left(k'+\frac{k^2}{k'}\right)J_1(ik'R)+\frac{2k^2}{\omega\rho}\tilde{p}(k,\omega)J_1(ikR)\right] \quad (4a)$$

# Derive Korteweg-Moens velocity

$$\frac{J_1(ikR)}{J_0(ikR)} = \frac{\tilde{\xi}_r^I(k)}{\tilde{\xi}_z^I(k)} = \frac{1}{\frac{hE}{R^2}}\left(\frac{i\omega^2\rho}{k}\right)$$

For kR<<1

$$\frac{J_1(ikR)}{J_0(ikR)} \sim \frac{1}{2}ikR$$

Using $c_0$=w/k within this equation

$$c_0^2 = \frac{Eh}{2\rho R}$$

Korteweg-Moens velocity

# Korteweg-Moens velocity

$$c_0^2 = \frac{Eh}{2\rho R} = \frac{Eh}{\rho d}$$

Korteweg-Moens velocity

Note that the speed of sound through a medium is given by

$$c^2 = \frac{E}{\rho}$$

The Korteweg-Moens velocity is like the speed of sound modified by the ratio of the wall thickness to vessel diameter. When wall thickness is zero the velocity is zero because the walls do not exist, so there is nothing to propagate along. Note this is the velocity along the vessel walls because it was obtained from Eq. 3a. When h~R, you get exactly the speed of sound, and it is referred to as acoustic waves.

# Wave equations derived from our Navier-Stokes equations for velocities

Integrating our Navier-Stokes equations over their cross-section converts velocities to volume flow rates, and imposing the boundary condition that fluid velocity and wall velocity must match at the boundary and that there is no viscosity, we find

$$\frac{\partial^2 p}{\partial t^2} = c_0^2 \frac{\partial^2 p}{\partial x^2}$$

$$\frac{\partial^2 \dot{Q}}{\partial t^2} = c_0^2 \frac{\partial^2 \dot{Q}}{\partial x^2}$$

Thus, we explicitly have wave equations for the pressure and the volume flow rate. Both of these have velocities determined by the Kortweg-Moens velocity for the case of inviscid flow.

# Wave velocity for our pressure waves

# More generally, the ratio of the actual phase velocity to the Korteweg-Moens velocity

$$\left(\frac{c}{c_0}\right)^2 \sim -\frac{J_2(ikr)}{J_0(ikr)}$$

You will do this as a homework problem. Note that the Korteweg-Moens velocity is a real number calculated from real physical quantities. However, the right side here is (in general) a complex number. Moreover, the right side depends on frequency, w, through k. Therefore, the traveling pressure wave has multiple frequency components and the notion of group velocity makes sense.

# Compare general phase velocity with Korteweg-Moens velocity

A wave with Korteweg-Moens velocity can be written as

$$e^{ik(c_0 t - z)}$$

Since all of the parameters here are real, this can be interpreted as a wave with an amplitude (constant coefficient out front) of 1, and a velocity of $c_0$

Since $c=c_1+ic_2$ is a complex number, the corresponding wave becomes

$$e^{ik(ct-z)} = e^{-kc_2 t} e^{ik(c_1 t - z)}$$

# Changes to amplitude and velocity

The amplitude (real part) of this wave is

$$e^{-kc_2 t}$$

Attenuation—change in amplitude of wave as it travels

The velocity of the wave is $c_1$ identified from the imaginary part, which we recall depends on the angular velocity w

$$e^{ik(c_1 t - z)}$$

Dispersion—velocity of wave varies with frequency of different components

# Impedance

# Impedance in Fourier space

This is now straightforward to calculate

$$\tilde{Z}(k) = \frac{\tilde{p}(k) J_0(ikR)}{\tilde{\dot{Q}}(k)} = \frac{\rho c_0^2}{\pi R^2 c}$$

This is a general expression for the impedance.

# Asymptotic Bessel function expansions

For small argument taking the first term of the Taylor series, we have

$$J_n(x) = \frac{1}{\Gamma(n+1)}\left(\frac{x}{2}\right)^n$$

For large argument, you can asymptotically solve differential equation or use integral representation to show

$$J_n(x) = \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)$$

# Investigate two limits

For kR<<1 (small vessels) (Laminar, Newtonian, Poiseuille flow)

$$\tilde{Z}(k) \sim \frac{8\mu l}{\pi R^4}$$

For kR>>1 (large vessels) (acoustic flow) traveling as if through air with no viscous drag

$$\tilde{Z}(k) \sim \frac{\rho c_0}{\pi R^2}$$

These two regimes give very different dependencies on the vessel radius and very different behaviors. We see both in the vascular system, as well as intermediate regimes. Most analytical studies assume one or the other. Numerical studies do not use formula but calculate directly from pressure and flow rate from Navier Stokes.