Modeling Vascular Networks with Applications

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Winter 2014 Quarter

Meeting time: Monday and Friday, 11:00am-12:50pm

Outline

1. Warm up for elastic vessel walls by solving equations when both the axial and radial components of the velocity are non-zero. This allows bulges and contractions in the radial movement of flow down the tube.
2. Understand equations for vessel walls
3. Look at equations for vessel walls as thin-wall approximation
4. Use boundary conditions to link vessel walls and fluid flow
5. Calculate volume flow rate, impedance, and Korteweg-Moens velocity in z-direction for case of massless walls

Movement of fluid in axial and radial directions

Movement of fluid in axial and radial directions

Velocity vector has two components

$$\vec{v} = (v_r, 0, v_z)$$

We still assume rotational symmetry, so no v_θ and no dependence of other velocity components on θ. Still avoiding real problem of turbulence by neglecting non-linearities, which we will continue to do. Our equations become

$$\rho \frac{dv_z}{dt} + \frac{dp}{dz} = \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) + \frac{\partial^2 v_z}{\partial z^2} \right]$$

$$\rho \frac{dv_r}{dt} + \frac{dp}{dr} = \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_r}{\partial r} \right) + \frac{\partial^2 v_r}{\partial z^2} - \frac{v_r}{r^2} \right]$$

Two more equations

Equation of continuity

$$\nabla \bullet \vec{v} = 0 \Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r v_r\right) = -\frac{\partial v_z}{\partial z}$$

Also, by taking divergence of our Navier-Stokes equations, we find

$$\nabla^2 p = 0 \Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) + \frac{\partial^2 p}{\partial z^2} = 0$$

This is Laplace's equation for the pressure, and it is one of the most well known PDEs that exists. Instead of separating off the pressure term as before, we can now solve Laplace's equation to use a more general method.

Solution to Laplace's equation

Using separation of variables and Fourier transforms as before, we obtain

$$p(r,z,t) = \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dk e^{i(\omega t - kz)} \tilde{p}(k,\omega) J_0(ikr)$$

Now add on an inhomogenous piece to cancel the pressure gradient for general solution

$$v_z(r,z,t) = \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dk e^{i(\omega t - kz)} [\hat{v}_z^H(k,\omega) J_0(ik' r) + \hat{v}_z^I(k,\omega) J_0(ikr)]$$

d/dz of this just brings down an *ik* that can be absorbed into the arbitrary function and this is *k* not *k*' because pressure just depends on k.

Using same tricks to find v_r

$$v_r(r,z,t) = \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dk e^{i(\omega t - kz)} [\hat{v}_r^H(k,\omega) J_1(ik' r) + \hat{v}_r^I(k,\omega) J_1(ikr)]$$

d/dr of this brings down an *ik* that can be absorbed into the arbitrary function and changes the Bessel function to zeroth order. Again, this is *k* not *k*' because pressure just depends on k.